$$\nabla^2 f - \frac{1}{c^2}\ddot{f} = 0 , \quad f\big|_{t=0} = p(x,y,z), \quad \dot{f}\big|_{t=0} = q(x,y,z) .$$

完整是

$$\frac{\partial^2 f(x,y,z,t)}{\partial x^2} + \frac{\partial^2 f(x,y,z,t)}{\partial y^2} + \frac{\partial^2 f(x,y,z,t)}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 f(x,y,z,t)}{\partial t^2} = 0 .$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$f \equiv f(x,y,z,t)$$

$$\ddot{f} \equiv \frac{\partial^2 f}{\partial t^2}$$

按 ∇^2 是埃尔米特型的,即存在特征函数并每个特征函数对应的特征值.

∇^2 的特征函数为 $\varphi_m(\vec{r}) = \frac{1}{\sqrt{\Omega}} e^{i\vec{k}\cdot\vec{r}}$ (其中 Ω 为标准化而引入).

在边长为 L 的立方体有限区域里,有: 再令 $-\nabla^2 \equiv H$, $\cancel{\nabla^2 f - \frac{1}{c^2}\ddot{f}}$

$$\lambda_m = c^2 k^2, \quad k_x = \frac{2\pi}{L} m_x , \quad k_y = \frac{2\pi}{L} m_y, \quad k_z = \frac{2\pi}{L} m_z, \qquad Hf + \ddot{f} = 0 .$$

$$|\vec{k}| = \frac{2\pi}{L}\sqrt{m_x^2 + m_y^2 + m_z^2} \quad ; \quad \vec{r} = (x,y,z)$$

则有:

$$f = \sum_m A_m(t)\,\varphi_m(\vec{r}) , \quad H\varphi_m = \lambda_m \varphi_m$$

$$\begin{aligned} Hf + \ddot{f} &= H\sum_m A_m(t)\varphi_m(\vec{r}) + \frac{\partial^2}{\partial t^2}\sum_m A_m(t)\varphi_m(\vec{r}) \\ &= \sum_m A_m(t)\, H\varphi(\vec{r}) + \sum_m \frac{\partial^2}{\partial t^2} A_m(t)\,\varphi_m(\vec{r}) \\ &= \sum_m A_m(t)\,\lambda_m \varphi_m(\vec{r}) + \sum_m \ddot{A}_m(t)\varphi_m(\vec{r}) \\ &= \sum_m \left[A_m(t)\lambda_m + \ddot{A}_m(t)\right]\varphi_m(\vec{r}) = 0 \end{aligned}$$

$$\Longrightarrow A_m(t) = a_m e^{i\sqrt{\lambda_m}t} + b_m e^{-i\sqrt{\lambda_m}t} = a_m\left[\cos(\sqrt{\lambda_m}t) + i\sin(\sqrt{\lambda_m}t)\right] + b_m\left[\cos(\sqrt{\lambda_m}t) - i\sin(\sqrt{\lambda_m}\right.$$

又.
$$\begin{aligned} f(\vec{r},0) &= \sum_m A_m(0)\,\varphi_m(x) \\ &= \sum_m \left(a_m e^{i\sqrt{\lambda_m}\cdot 0} + b_m e^{-i\sqrt{\lambda_m}\cdot 0}\right)\varphi_m(\vec{r}) \\ &= \sum_m (a_m + b_m)\,\varphi_m(\vec{r}) \end{aligned}$$

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$$f'(\vec{r},0) = \sum_m i\sqrt{\lambda_m}\,(a_m - b_m)\,\varphi_m(\vec{r})$$

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$$\cancel{\int f(\vec{r},0)\cdot\overline{\varphi_m(\vec{r})} = \int\sum_m (a_m+b_m)\varphi_m(\vec{r})\cdot\overline{\varphi_m(\vec{r})}\,d\tau^3 = \int a}$$

$$\int f(\vec{r},0)\cdot\overline{\varphi_m(\vec{r}')}\,d\vec{r}' = \int\sum_m (a_m+b_m)\varphi_m(\vec{r}')\,\overline{\varphi_m(\vec{r}')}\,d\vec{r}' = a_m+b_m$$

$$\int f'(\vec{r},0)\cdot\overline{\varphi_m(\vec{r}')}\,d\vec{r}' = \int\sum_m (a_m-b_m)\varphi_m(\vec{r}')\,\overline{\varphi_m(\vec{r}')}\,d\vec{r}' = a_m-b_m$$

$$\Rightarrow a_m = \frac{1}{2}\left[\int f(\vec{r},0)\varphi_m^*(\vec{r})\,d\vec{r}' + \frac{1}{i\sqrt{\lambda_m}}\int f'(\vec{r},0)\overline{\varphi_m(\vec{r}')}\,d\vec{r}'\right]$$

$$b_m = \frac{1}{2}\left[\int f(\vec{r},0)\varphi_m(\vec{r}')\,d\vec{r}' - \frac{1}{i\sqrt{\lambda_m}}\int f'(\vec{r},0)\overline{\varphi_m(\vec{r}')}\,d\vec{r}'\right]$$

$$A_m(t) = \int\cos(\sqrt{\lambda_m}t)\varphi_m(\vec{r}')f(\vec{r},0)\,d\vec{r}' + \int\frac{\sin(\sqrt{\lambda_m}t)}{\sqrt{\lambda_m}}f'(\vec{r},0)\overline{\varphi_m(\vec{r}')}\,d\vec{r}'$$

$$f = \sum_m A_m(t)\,\varphi_m(\vec{r})$$

$$= \int\underbrace{\sum_m\cos(\sqrt{\lambda_m}t)\varphi_m(\vec{r}')\,\cancel{f(\vec{r}',0)}\,\varphi_m(\vec{r})}_{\equiv G_1(t,\vec{r}',\vec{r})}\,d\vec{r}' + \int\underbrace{\sum_m\frac{\sin(\sqrt{\lambda_m}t)}{\sqrt{\lambda_m}}\,\cancel{f(\vec{r}',0)}\,\overline{\varphi_m(\vec{r}')}\,\varphi_m(\vec{r})}_{\equiv G_2(t,\vec{r}',\vec{r})}\,d\vec{r}'$$

易知 $\dfrac{\partial G_2(t,\vec{r}',\vec{r})}{\partial t} = G_1(t,\vec{r}',\vec{r})$

$$G_2(t,\vec{r},\vec{r}) = \sum_m\frac{\sin(\lambda_m t)}{\sqrt{\lambda_m}}\frac{1}{\sqrt{\Omega}}\cancel{e^{i\vec{k}(\vec{r}-\vec{r}')}}\,e^{i\vec{k}\cdot\vec{r}}\cdot\frac{1}{\sqrt{\Omega}}e^{-i\vec{k}\cdot\vec{r}'}$$

$$= \sum_{m_x,m_y,m_z}\frac{1}{\Omega}\frac{\sin(ckt)}{ck}e^{i\vec{k}\cdot(\vec{r}-\vec{r}')}$$

2/3.

$L\to\infty$ 时（边长为 L 的正方形趋于自由空间）.

$$G_2(t,\vec{r}',\vec{r}) = \frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty}\frac{\sin(ckt)}{ck}e^{i\vec{k}\cdot(\vec{r}-\vec{r}')}\,dk^3.$$

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$$= \frac{1}{(2\pi)^3}\int_0^{2\pi}d\varphi\int_0^{\pi}d\theta\int dk\,e^{ik\rho\cos\theta}\frac{e^{ikct}-e^{-ikct}}{2ikc}k^2\sin\theta \quad \text{—— } \vec{\rho}=\vec{r}-\vec{r}',\ \theta=\angle(\vec{k},\vec{\rho})$$

$dk^3 = k^2\sin\theta\,dk\,d\theta\,d\varphi$

$$= \frac{1}{4\pi^2}\int_0^{\infty}\frac{e^{ik\rho}-e^{-ik\rho}}{ik\rho}\cdot\frac{e^{ikct}-e^{-ikct}}{2ikc}\cdot k^2\,dk$$

$$= -\frac{1}{8\pi^2c\rho}\int[e^{ik(\rho+ct)}-e^{ik(\rho-ct)}]\,dk \quad \text{—— } \frac{1}{2\pi}\int_{-\infty}^{+\infty}e^{ikx}\,dk = \delta(x).$$

$$= \frac{1}{4\pi c\rho}[\delta(\rho-ct)-\delta(\rho+ct)]$$

$$= \frac{1}{4\pi c|\vec{r}-\vec{r}'|}[\delta(|\vec{r}-\vec{r}'|+ct)-\delta(|\vec{r}-\vec{r}'|-ct)\} = \frac{1}{4\pi c|\vec{r}-\vec{r}'|}\delta(|\vec{r}-\vec{r}'|-ct)$$

—— $t>0$.

$$f = \sum_m A_m(t)\,\varphi_m(\vec r)$$

$$= \int G_1(t,\vec r',\vec r)\, f(\vec r',0)\, \cancel{\varphi_m(\vec r)}\, d\vec r' + \int G_2(t,\vec r',\vec r)\, f'(\vec r,0)\, d\vec r'$$

$$= \frac{\partial}{\partial t}\int_V \frac{1}{4\pi c|\vec r-\vec r'|}\,\delta(|\vec r-\vec r'|-ct)\,\overset{f(\vec r,0)}{d\vec r'} + \int_V \frac{1}{4\pi c|\vec r-\vec r'|}\,\delta(|\vec r-\vec r'|-ct)\, f'(\vec r,0)\, d\vec r'$$

$$\cancel{= \left(\frac{\partial}{\partial t}\iint_{S_{r=at}} f(\vec r,0)\,dS + \iint_{S_{r=at}} f'(\vec r,0)\,dS\right)\frac{1}{4\pi c}\cdot\frac{1}{at}}$$

$d\vec r'$ 仅在 $\cancel{r=at}$ $r=ct$ 的球上有效. 根据 $\delta(|\vec r-\vec r'|-ct)$ 的性质.

$$= \frac{1}{4\pi c}\left(\frac{\partial}{\partial t}\iint_{S_{r=ct}} \frac{f(\vec r,0)}{ct}\,ds + \iint_{S_{r=ct}} \frac{f'(\vec r,0)}{ct}\,ds\right)$$

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